

# An exploration of normalish subgroups of R. Thompson's groups $F$ and $T$

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March 8, 2016

## Abstract

In this short note, we show that R. Thompson's group  $F$  admits a normalish amenable subgroup, and that the standard copy of  $F$  in R. Thompson's group  $T$  is normalish in  $T$ . We further conjecture that if  $F$  is non-amenable, then  $T$  does not admit a normalish amenable subgroup, and therefore that the reduced  $C^*$  algebra of  $T$  is in fact simple in that case.

**Keywords:** *R. Thompson Groups, Normalish Subgroups,  $C^*$ -simplicity*

## 1 Introduction

In [7], the authors show that if R. Thompson's group  $T$  has a simple reduced  $C^*$  algebra ( $T$  is  $C^*$ -simple), then  $F$  is non-amenable. Later, in [1] it is shown that the Kesten Test (commonly used to detect Powers' Criterion for  $C^*$ -simplicity) cannot be used in the case of the group  $T$ . This and other anecdotal evidence has lead some to speculate that  $T$  might fail to be  $C^*$ -simple despite the fact that  $T$  is a group with trace. If true,  $T$  would provide a new and interesting example of this rare phenomenon (see [6] for Adrien Le Boudec's examples of such groups).

The paper [3] provides a new test for  $C^*$ -simplicity for a group. Namely, if a group admits no *normalish* amenable subgroups, then it is  $C^*$ -simple. A group  $M$  is *normalish in a group  $G$*  if and only if given any finite set  $K := \{k_1, k_2, \dots, k_m\} \subset G$  we have

$$\cap_{i=1}^m M^{k_i} \neq \{1_G\}.$$

Focussing now on the R. Thompson groups, consider  $S^1 = \mathbb{R}/\mathbb{Z}$  (we use the parameterisation provided by the map  $t \mapsto e^{2\pi it}$ ). The *standard R. Thompson's groups*  $F < T$  are groups of orientation-preserving piecewise-linear homeomorphisms of  $S^1$  which preserve the dyadic rationals and which admit at most finitely many breaks in slope, appearing only over the dyadic rationals  $\{\frac{a}{2^k} \mid a, k \in \mathbb{Z}\}$ , and with all slopes of affine components being integral powers of 2. In this case,  $T$  is the full group of such homeomorphisms, whereas  $F$  is the subgroup of  $T$  which stabilises the point 0 under

the natural action of  $T$  on  $S^1$ . See [5] for a survey of the important R. Thompson groups.

Our first result has bearing on R. Thompson's group  $F$ .

**Theorem 1.** *R. Thompson's group  $F$  admits a normalish amenable subgroup.*

Our second observation is that the canonical version of  $F$  in  $T$  is a normalish subgroup in R. Thompson's group  $T$ .

**Remark 2.** *For the standard R. Thompson groups  $F < T$ , we have that  $F$  is normalish in  $T$ .*

In particular, in the case that  $F$  is amenable, then  $T$  admits a normalish amenable subgroup. This of course agrees with the Haagerup-Olesen result that if  $T$  is  $C^*$ -simple, then  $F$  is non-amenable.

*Acknowledgements:* We would like to thank Emmanuel Breuillard and Kate Juschenko for interesting discussions of the  $C^*$ -simplicity of the groups  $F$  and  $T$ , where their questions and discussions motivated the creation of this note.

## 1.1 Further explorations

It is of abiding interest whether the converse of the Haagerup-Olesen result is also true. As mentioned before, some researchers have noted that the result in [1] provides a first indication that  $T$  may not be  $C^*$ -simple even if  $F$  is non-amenable. We do not take this view.

The set  $\mathcal{X}$  of subgroups of  $T$  which admit no embedded copies of R. Thompson's group  $F$ , and no embedded non-abelian free subgroups, is a very interesting and complex set. Two key results which may be of assistance in understanding constraints on the elements of groups in the set  $\mathcal{X}$  are Brin's Ubiquity Theorem, and Lemma 1.9 of [2]. To state Brin's Ubiquity Theorem, we require Brin's notion of *orbital*. We say a group  $H \leq \text{PL}_o(I)$  has an orbital  $(a, b)$  if  $(a, b)$  is an open interval in  $I := [0, 1]$  and the interval  $(a, b)$  is a component of support of the action of  $H$  on  $I$ . Likewise, if  $g \in H$ , we say an interval  $(c, d)$  is an orbital of  $g$  if it is an orbital of  $\langle g \rangle$ . If  $H$  has an orbital  $(a, b)$  and  $c \in (a, b)$ , and  $g \in H$  has an orbital of the form  $(a, c)$  or  $(c, b)$  then we say  $g$  approaches  $a$  (or respectively  $b$ ) in  $(a, b)$ .

**Theorem 3** (Brin's Ubiquity). *Let  $H$  be a subgroup of  $\text{PL}_o(I)$ . Assume that  $H$  has an orbital  $(a, b)$  and that some element of  $H$  approaches one end  $a$  or  $b$  in  $(a, b)$  but not the other. Then  $H$  contains a subgroup isomorphic to  $F$ .*

A form of the contrapositive of Lemma 1.9 of [2] is the following lemma.

**Lemma 4.** *Let  $1 < k \in \mathbb{N}$  and suppose  $\mathcal{F} := \{g_1, g_2, \dots, g_k\}$  is a set of orientation-preserving homeomorphisms of  $S^1$  each of which admits a non-trivial fixed set, and suppose further that*

$$\bigcap_{1 \leq i \leq k} \text{Fix}(g_i) = \emptyset,$$

*then  $\langle g_1, g_2, \dots, g_k \rangle$  contains non-abelian free subgroups.*

Together, these results imply that for any  $H \in \mathcal{X}$  the elements of  $H$  admit very strong conditions on how their components of support can overlap. While we will not investigate this further here, we note that our proof that  $F$  is normalish in  $T$  does not appear to translate directly to apply to such a group  $H$ , and so we conjecture the following.

**Conjecture 5.** *Every normalish subgroup of  $T$  contains embedded copies of R. Thompson's group  $F$  or of non-abelian free subgroups.*

## 2 Normalish subgroups exist

In this section we show our Theorem 1 and Remark 2. First though, we recall some notation.

Let  $g \in F \cup T$ , and recall the definition

$$\text{Supp}(g) := \{x \in (0, 1) \mid x \cdot b \neq x\}.$$

We are now ready to prove Theorem 1.

*Proof.* Consider the standard generator  $x_0$  of  $F$ , and another function  $b \in F$  so that  $\text{Supp}(b) = (1/4, 1/2)$ . Since  $1/4 \cdot x_0 = 1/2$ , we have that  $W = \langle x_0, b \rangle \cong \mathbb{Z} \wr \mathbb{Z}$ , noting that this is a standard construction. We observe that the base group  $D$  of  $W$  is generated by  $X := \{b^{x_0^k} \mid k \in \mathbb{Z}\}$  which is all of the conjugates of  $b$  by  $x_0$ . Further note that

$$D \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}$$

as the generators in the set  $X$  are pairwise disjoint.

We now observe that as any element of  $F$  is linear near some neighbourhood of 0, with slope some value  $2^k$ , we have that  $|X^c \cap X| = \infty$  for any conjugator  $c$ , since many of these generators will be taken to each other in a small neighbourhood of 0 by the conjugation action of the first linear part of  $c$ .

Therefore  $D$  is a normalish amenable subgroup of  $F$ .  $\square$

Similarly, we will now show that the standard version of R. Thompson's group  $F$  in  $T$  is normalish in  $T$ .

*Proof.* Let  $K = \{k_1, k_2, \dots, k_m\} \subset T$  and consider the set

$$Y := \{F^{k_i} \mid 1 \leq i \leq m\}$$

which consists of subgroups of  $T$  which are conjugates of  $F$ .

From the definition of  $F$  given in the introduction,  $F$  contains all orientation-preserving dyadic pl-homeomorphisms of  $[0, 1]$ ; these are the homeomorphisms which fix 0 and 1 (identified in the circle  $S^1$ ), having (only finitely many) breaks in slope, all of which are occurring at dyadic rationals, and where all slopes of affine components are integral powers of 2. In particular, for any dyadic interval  $(\frac{a}{2^j}, \frac{b}{2^k})$ , consider the set  $B_{(\frac{a}{2^j}, \frac{b}{2^k})}$  of all elements of  $F$  supported exactly on  $(\frac{a}{2^j}, \frac{b}{2^k})$ , which set is commonly

known to generate a subgroup of  $F$  isomorphic to  $F$  (this follows easily from Cannon, Floyd and Parry's survey [5]). It is now immediately the case that the intersection

$$\mathcal{J} := \cap_{1 \leq i \leq m} F^{k_i}$$

is an infinite set. This is easy to see, since it contains any such set  $B_{(\frac{a}{2^j}, \frac{b}{2^k})}$  where the interval  $(\frac{a}{2^j}, \frac{b}{2^k})$  is disjoint from the set

$$\{0 \cdot k_i \mid 1 \leq i \leq m\} \cup \{0\}.$$

□

We comment that there are other arguments for our Remark 2 which might be easier than the above, but, one should be careful:  $T$  admits subgroups with global fixed point set not empty, which are not conjugate in  $T$  to subgroups of the canonical embedding of  $F$  in  $T$ .

## References

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